

# Gauge invariant observables from Takahashi-Tanimoto scalar solutions in open string field theory

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## Abstract

Using Maccaferri's formula, we derive new wedge based solutions of open string field theory. The solutions are gauge equivalent to the Takahashi-Tanimoto scalar solutions. The classical action and the gauge invariant overlap are evaluated analytically. We find a perturbative vacuum solution whose gauge invariant observables vanish. We also identify a tachyon vacuum solution whose gauge invariant observables are identical to those of the Erler-Schnabl solution.

## 1 Introduction

One of major motivations for studying string field theory (SFT) is to understand its background independence. In SFT, a string background specified by a specific conformal field theory (CFT) corresponds to a classical solution of the equation of motion. Therefore, SFT may offer an unified framework to derive all possible string vacua from the unique action. One can study relations between different vacua if corresponding classical solutions are available.

Recently, there has been remarkable progress [1, 2, 3] for solutions which describe marginally deformed boundary conformal field theory (BCFT). The novelty of them is that they employ the Takahashi-Tanimoto (TT) identity based solution [4] as a regularization for the singular OPE between marginal currents on the boundary. There, the singularities are nicely regularized by an integral of the marginal current extended to the bulk. In [1], marginally deformed tachyon vacuum has been obtained. [2] has derived the gauge invariant overlap (or Ellwood invariant) for identity based Takahashi-Tanimoto solution as a difference of the overlap between wedge based solutions. In [3], Maccaferri has constructed a wedge based solution for marginal deformation. In his work, the formula which expresses a gauge invariant observable<sup>1</sup> as a difference between that of Erler-Schnabl solution [8] evaluated on the original background and marginally deformed background plays a central role. Using this formula, the gauge invariant observables for marginal deformation have been evaluated analytically. Remarkable feature of [3] is that the formulas can also be applicable to arbitrary backgrounds as long as a “seed”  $\Phi$  satisfies the equation of motion.

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<sup>1</sup>In this paper, we discuss two well-known gauge invariant observables, i.e., the action and the gauge invariant overlap [5, 6, 7].

As an immediate application of Maccaferri's formula, we will consider the TT scalar solution [4] in this paper. As similar to the TT marginal solution, this solution is identity based. In sliver frame, the solution is defined by line integrals of the BRST current  $j_B(z)$  and conformal ghost  $c(z)$  along the imaginary axis convoluted by appropriate functions. Explicitly,

$$\Phi = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} (f(z)j_B(z) + g(z)c(z)). \quad (1.1)$$

Here  $f(z)$  and  $g(z)$  is not independent and their relation follows from the equation of motion. Much works have been devoted [4, ] to possible choice of  $f(z)$ , and it has been conjectured that particular choice give rise to the tachyon vacuum. However, direct derivation of the gauge invariant observables has not yet available since the solution is identity based, i.e., it is defined on an infinitesimally thin strip in sliver frame. In this paper, we derive gauge invariant observables, which have been only studied numerically [15, ] by extending the TT solution to a wedge based solution which allows analytic treatment on the world sheet.

This paper is organized as follows. Section 2 introduces the formulas of [3]. In section 3, we apply the formulas to well-known identity based solutions in  $KBc$  subalgebra [18, 19]. Although these solutions are well understood, we present it for demonstration purpose. Section 4 deals with the TT scalar solutions, which is of our main interest. We conclude in section 5 and give discussions.

*Note added:* While revising this manuscript, we found two papers [20, 21] which deal with similar subjects with our paper.

## 2 Maccaferri's formulas

We begin with a review of the formalism developed by Maccaferri [3], on which we heavily rely in this paper. The formalism allows us to map an identity based solution to a wedge based solution whose gauge invariant observable is defined by a CFT correlator on a cylinder with nonzero width. Let  $\Phi$  be a "seed" solution of the equation of motion  $Q_B\Phi + \Phi^2 = 0$ . Then, a new wedge based solution is given by,

$$\Psi = \frac{1}{1+K} \left( \Phi - \Phi \frac{B}{1+K'} \Phi \right), \quad (2.2)$$

where  $K$  and  $B$  are familiar elements of the  $KBc$  subalgebra and  $K'$  is the deformed generator defined by

$$\begin{aligned} K' &= Q_{\Phi\Phi}B \\ &\equiv Q_BB + \{B, \Phi\}. \end{aligned} \quad (2.3)$$

Note that  $K'$  is an element of deformed algebra,

$$Q_{\Phi\Phi}c = cK'c, \quad Q_{\Phi\Phi}B = K', \quad Q_{\Phi\Phi}K' = 0, \quad \{c, B\} = 1. \quad (2.4)$$

The solution (2.2) is gauge equivalent to  $\Phi$  since it can be written as a  $\Phi$  dependent gauge transformation

$$\Psi = (1 + A\Phi)Q_B \left( \frac{1}{1 + A\Phi} \right) + (1 + A\Phi)\Phi \frac{1}{1 + A\Phi}, \quad (2.5)$$

where  $A = B/(1 + K)$ . Notable feature of this prescription is that gauge invariant observables are expressed as a difference between those of the Erler-Schnabl solution [8] defined on perturbative vacuum and deformed background by  $\Phi$ , respectively. For example, the gauge invariant overlap for closed string vertex operator  $V$  is given by <sup>2</sup>

$$\begin{aligned}\mathrm{Tr}_V[\Psi] &= \mathrm{Tr}_V[\Psi_{ES,K}] - \mathrm{Tr}_V[\Psi_{ES,K'}] \\ &= \mathrm{Tr}_V \left[ c \frac{1}{1+K} \right] - \mathrm{Tr}_V \left[ c \frac{1}{1+K'} \right],\end{aligned}\tag{2.6}$$

where  $\Psi_{ES,K}$  is the original Erler-Schnabl solution  $\frac{1}{1+K}(c + Q_B(Bc))$  and  $\Psi_{ES,K'}$  is that defined on the deformed background. Quite similarly, the action is also expressed as a difference

$$\begin{aligned}S[\Psi] &= S[\Psi_{ES,K}] - S[\Psi_{ES,K'}] \\ &= -\frac{1}{6}\mathrm{Tr} \left[ \frac{1}{1+K} c Q_B \left( \frac{1}{1+K} c \right) \right] + \frac{1}{6}\mathrm{Tr} \left[ \frac{1}{1+K'} c Q_{\Phi\Phi} \left( \frac{1}{1+K'} c \right) \right].\end{aligned}\tag{2.7}$$

As claimed in [3], derivations of (2.6) and (2.7) only require algebraic manipulation in the traces in terms of the  $KBc$  subalgebra and the equation motion of  $\Phi$ , therefore do not depend on any details of  $\Phi$ . While  $\Phi$  can be arbitrary solution, we will consider identity based  $\Phi$ .

### 3 Gauge invariant observables for identity based $KBc$ solutions

Next, we apply the Maccaferri's formulas to the identity based solutions in  $KBc$  subalgebra [18, 19] to evaluate gauge invariant observables. We refer the classification given in [19] where the author identified three kinds of gauge orbits, perturbative vacuum, tachyon vacuum, and the MNT ghost brane. In following sections, we deal with perturbative vacuum and tachyon vacuum which are relevant to later discussions for the TT solution.

#### 3.1 Perturbative vacuum

The ‘‘perturbative vacuum’’ solution is the well-known BRST exact string field,

$$\Phi = \lambda cKBc \quad (\lambda \neq -1),\tag{3.8}$$

where  $\lambda$  is a real constant<sup>3</sup>. This solution can be written in pure gauge form  $\Phi = UQ_B U^{-1}$  where

$$U = 1 + \lambda cB, \quad U^{-1} = 1 - \frac{\lambda}{1+\lambda} cB.\tag{3.9}$$

The deformed generator for this solution is found to be

$$K' = K + \lambda(cKB + BKc).\tag{3.10}$$

It is not difficult to see that a  $n$ -th power of  $K'$  becomes

$$K'^n = (1 + \lambda)^{n-1} \{K^n + \lambda(BK^n c + cK^n B)\}.\tag{3.11}$$

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<sup>2</sup>Here  $\mathrm{Tr}_V[\Psi] = \mathrm{Tr}[V\Psi]$  and  $V = c(i\infty)c(-i\infty)V_X(i\infty, -i\infty)$ , where  $V_X$  is a weight  $(1, 1)$  vertex operator in matter sector.

<sup>3</sup> $\lambda = -1$  is excluded because we cannot construct the similarity transformation discussed in this section. It also should be noted that this string field is a part of the MNT ghost brane solution [19] which is not gauge equivalent to the perturbative vacuum.

Therefore, a function of  $K'$  defined by a power series also satisfies

$$f(K') = \frac{1}{1+\lambda} \{f(K_\lambda) + \lambda(Bf(K_\lambda)c + cf(K_\lambda)B)\} \quad (3.12)$$

where  $K_\lambda = (1+\lambda)K$ . The formulas (3.11) and (3.12) can also be explained by a gauge transformation  $U$  since it acts on  $K$ ,  $B$  and  $c$  in very simple manner,

$$UKU^{-1} = \frac{K'}{1+\lambda}, \quad UcU^{-1} = (1+\lambda)c, \quad UBU^{-1} = \frac{B}{1+\lambda}. \quad (3.13)$$

In order to evaluate the gauge invariant overlap, we apply (3.13) to the last term of (2.6) to obtain

$$\text{Tr} \left[ V \frac{1}{1+K'} c \right] = \text{Tr} \left[ VU \frac{1}{1+K_\lambda} U^{-1} c U U^{-1} \right] \quad (3.14)$$

$$= (1+\lambda)^{-1} \text{Tr} \left[ U^{-1} VU \frac{1}{1+K_\lambda} c \right], \quad (3.15)$$

where  $K_\lambda = (1+\lambda)K$ . The quantity  $U^{-1}VU = U^{-1}c(i\infty)c(-i\infty)V_XU$  can be calculated by using  $[cB, c(i\infty)c(-i\infty)] = cc(-i\infty) - cc(+i\infty)$ :

$$U^{-1}VU = V - \frac{\lambda}{1+\lambda} c(c(-i\infty) - c(i\infty))V_X. \quad (3.16)$$

The second term of (3.16) does not contribute to the trace (3.15) because of it collides with another  $c$  in the trace. Therefore the trace reduces to

$$(1+\lambda)^{-1} \text{Tr} \left[ \frac{1}{1+K_\lambda} c \right] = \text{Tr} \left[ \frac{1}{1+K} c \right]. \quad (3.17)$$

Here the  $(1+\lambda)^{-1}$  factor in front of the trace and the  $\lambda$  dependence of  $K_\lambda$  in the left hand side of (3.17) are absorbed into scaling of the correlation function on the cylinder. The obtained trace is just the gauge invariant observable for the Erler-Schnabl solution. This cancels first term of (2.6), and then the gauge invariant observable  $\text{Tr}_V[\Psi]$  vanishes as expected.

The action can be evaluated similarly. Using (3.13), the trace in the second term of (2.7) is evaluated as

$$(1+\lambda)^{-3} \text{Tr} \left[ c \frac{1}{1+K_\lambda} c K_\lambda c \frac{1}{1+K_\lambda} \right]. \quad (3.18)$$

Again, the  $1+\lambda$  factors in the trace cancels after scaling of the correlation function on the cylinder. Again, this is exactly same as the first term of (2.7). Therefore we obtain vanishing action.

Although we apply Maccaferri's formulas for a demonstration, our result can be confirmed by direct evaluation of the original solution  $\Psi$ . Plugging  $\Phi = \lambda c K B c$  into (2.5), it is soon realized that the entire solution is nothing but a non-real form of the Okawa solution [22],

$$\Psi = F^2 c \frac{K}{1-F^2} B c, \quad (3.19)$$

where  $F^2$  is

$$F^2 = \frac{\lambda}{1+\lambda} \times \frac{1}{1+K}. \quad (3.20)$$

The homotopy operator for this solution

$$A = \frac{1}{1+\lambda} \frac{1+\lambda+K}{K(1+K)} \quad (3.21)$$

indicates that the BRST cohomology is nontrivial due to  $1/K$  dependence in (3.21). Clearly this is a wedge based perturbative vacuum solution according to the classification given in [23].

### 3.2 Tachyon vacuum

Next, we consider an identity based solution

$$\Phi = c - Kc \quad (3.22)$$

which is gauge equivalent to the wedge based tachyon vacuum solution. Application of the Maccaferri's formulas to (3.22) is straightforward since  $K'$  becomes constant as one can easily verify from (2.3). In this case,  $K' = 1$  and the piece of the solution which is relevant to the gauge invariant observables is given by

$$\frac{1}{1+K'}c = \frac{1}{2}c. \quad (3.23)$$

The closed string overlap is given by

$$\text{Tr}_V[\Psi] = \text{Tr}_V[\Psi_{ES,K}] - \frac{1}{2}\text{Tr}_V[c]. \quad (3.24)$$

The last term in (3.24) is a trace of identity based string field therefore cannot be evaluated directly. We claim that this trace vanishes as explained below. While there could be several way to prove this, we follow the strategy employed to the case of the marginal deformation [24, 1, 2, 3] in which a contribution from the deformation  $\delta K = K' - K$  is expressed as a path ordered exponential. More precisely, we apply the Schwinger parametrization

$$\begin{aligned} \frac{1}{1+K'} &= \frac{1}{1+K+(-K+1)} \\ &= \int_0^\infty dt e^{t(K-1)} e^{-t(1+K)}. \end{aligned} \quad (3.25)$$

to the last term of (3.24). We regard  $e^{t(K-1)}$  as deformation for the original background. In present case,  $\delta K = 1 - K$  commutes with  $K$ , so path ordering is not necessary. Expanding  $e^{t(K-1)}$  in  $K$  and regard it as a sum of differentials on  $e^{-t(1+K)}$ , we have

$$\frac{1}{1+K'} = \lim_{s \rightarrow 1} \int_0^\infty dt e^{-2t} e^{-\partial_s} e^{-stK}. \quad (3.26)$$

Then, second term of (3.24) can be evaluated as

$$\begin{aligned} \int_0^\infty dt e^{-2t} \frac{1}{1+K'} &= \lim_{s \rightarrow 1} \int_0^\infty dt e^{-2t} e^{-\partial_s}(st) \text{Tr}_V[c] \\ &= \text{Tr}_V[c] \lim_{s \rightarrow 1} (s-1) \int_0^\infty dt e^{-2t} t \\ &= 0, \end{aligned} \quad (3.27)$$

where we have scaled the correlation function on the cylinder in first line of (3.27). Then, only the first term of (3.24) remains to give a same value as that of the Erler-Schnabl solution just expected.

The classical action can be evaluated more straightforwardly by applying  $K' = 1$  to (2.7). In this case, last term in (2.7) vanishes trivially since it is just proportional to  $\text{Tr}[cK'c] = \text{Tr}[c^3]^4$ . Therefore the value of the classical action also coincides with that of Erler-Schnabl solution.

Again similar to the end of the former section, our result is confirmed by a form of the whole solution. Plugging  $\Phi = c - Kc$  into (2.5), one soon realize that the entire solution is nothing but a non-real form of the Okawa solution [22],

$$\Psi = F^2 c \frac{K}{1 - F^2} Bc, \quad (3.28)$$

where  $F^2 = (1 - K)/(1 + K)$  is just a product of  $1 - K$  and  $1/(1 + K)$  which define two Okawa solutions  $c - Kc$  and  $1/(1 + K)c(1 + K)Bc$ . According to the classification given in [25], this solution corresponds to tachyon vacuum since homotopy operator turns out to be  $2B/(1 + K)$  which is proportional to that of Erler-Schnabl solution. Hence BRST cohomology for this solution is trivial.

## 4 Takahashi-Tanimoto scalar solution

We would like to describe the Takahashi-Tanimoto (TT) scalar solution [4] along the line with [26]. The solution is written as

$$\Psi = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \left( f(z) j_B(z) + g(z) c(z) \right). \quad (4.29)$$

Here we follow the convention of [3], where  $c(z)$  and  $j_B(z)$  are identity based string fields rather than conformal fields in sliver frame [27]. For example, the string field  $c(z)$  is defined by

$$c(z) = e^{zK} c e^{-zK}. \quad (4.30)$$

This description is very useful since  $c(z)$  obeys OPE like equation and actually becomes a conformal field once inserted into the world sheet. The  $z$  integration runs along the imaginary axis placed at the center of the infinitely thin vertical strip. The functions  $f(z)$  and  $g(z)$  should be defined on the imaginary axis. They are not dependent and fixed by the equation of motion. In order to solve equation of motion for (4.29), we need to evaluate a product of line integrals. As shown in [4, 3], one of the line integral in the product can be converted into a contour integral around other operator in line integral. For example, a cross terms of  $\int dz f(z) j_B(z)$  and  $\int dw g(w) c(w)$  is evaluated as follows:

$$\begin{aligned} \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} f(z) \int_{-i\infty}^{i\infty} \frac{dw}{2\pi i} g(w) \{j_B(z), c(w)\} &= \oint \frac{dz}{2\pi i} f(z) \int_{-i\infty}^{i\infty} \frac{dw}{2\pi i} g(w) j_B(z) c(w) \\ &= \oint \frac{dz}{2\pi i} f(z) \int_{-i\infty}^{i\infty} \frac{dw}{2\pi i} \frac{c(w) \partial c(w)}{z - w} \\ &= \int_{-i\infty}^{i\infty} \frac{dw}{2\pi i} f(w) g(w) c \partial c(w) \end{aligned} \quad (4.31)$$

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<sup>4</sup>Although this is identity based, we respect the algebraic rule  $\text{Tr}[c^n] = 0$ .

In the last line, we use the OPE

$$j_B(z)c(w) \sim \frac{1}{z-w}c\partial c(w). \quad (4.32)$$

Let us introduce a shorthand notation

$$f \cdot j_B = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} f(z) j_B(z), \quad g \cdot c = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} g(z) c(z). \quad (4.33)$$

Then, the result (4.31) is simply stated as

$$\{f \cdot j_B, g \cdot c\} = (fg) \cdot c\partial c. \quad (4.34)$$

In order to derive equation of motion, OPE of between two  $j_B$ s

$$j_B(z)j_B(w) \sim -\frac{4}{(z-w)^3}c\partial c(w) - \frac{2}{(z-w)^2}c\partial^2 c(w) \quad (4.35)$$

is also required. Then another formula follows,

$$\{f_1 \cdot j_B, f_2 \cdot j_B\} = 2(f'_1 f'_2) \cdot c\partial c, \quad (4.36)$$

where primes denote  $z$  derivative.  $Q_B(f \cdot j_B)$  and  $Q_B(g \cdot c)$  can also be evaluated by regarding  $Q_B \sim 1 \cdot j_B$ . Then the right hand side of the equation motion  $Q_B\Psi + \Psi^2 = 0$  is reduced to a line integral of  $c\partial c$  multiplied by

$$g(z) + f(z)g(z) + f'(z)^2. \quad (4.37)$$

Therefore the equation of motion holds if  $f(z)$  and  $g(z)$  satisfy

$$g(z) + f(z)g(z) + f'(z)^2 = 0. \quad (4.38)$$

For given  $f(z)$ ,  $g(z)$  is easily solved as

$$g(z) = -\frac{f'(z)^2}{1+f(z)}. \quad (4.39)$$

In original TT paper [4], solutions for given  $f(z)$  have been considered. On the other hand, (4.38) can also be solved for given  $g(z)$ . In order to do this, we write  $f(z)$  as

$$f(z) = -\frac{1}{4}\omega(z)^2 - 1. \quad (4.40)$$

Then, (4.38) becomes much simpler

$$-g(z) + \omega'(z)^2 = 0 \quad (4.41)$$

which can be easily integrated as

$$\omega(z) = \int_{z_0}^z dz' \sqrt{g(z')}. \quad (4.42)$$

Then  $f(z)$  is finally given by

$$f(z) = -\frac{1}{4} \left( \int_{z_0}^z dz' \sqrt{g(z')} \right)^2 - 1. \quad (4.43)$$

Note that  $\omega(z)$  is defined up to an integration constant, which will be fixed later.

Let us now turn to the deformed generator  $K' = K + \{B, \Psi\}$ , which is of our main interest, since it is required for the Maccaferri's formulas (2.6) and (2.7) in order to evaluate the gauge invariant observables. From the OPEs

$$j_B(z)b(w) \sim \frac{3}{(z-w)^3} + \frac{j_g(w)}{(z-w)^2} + \frac{T(w)}{z-w}, \quad (4.44)$$

$$b(z)c(w) \sim \frac{1}{z-w}, \quad (4.45)$$

one can derive

$$\{h \cdot b, f \cdot j_B\} = (fh) \cdot T + (f'h) \cdot j_g + \frac{3}{2}f''g \quad (4.46)$$

$$\{h \cdot b, f \cdot c\} = fg. \quad (4.47)$$

Here a term which is not convoluted with operator denotes a constant obtained by a line integral. For example,

$$fg = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} f(z)g(z). \quad (4.48)$$

A sum of (4.46), (4.47) for  $h(z) = 1$  and the original  $K$  corresponds to  $K'$ , whose expression is

$$K' = (1 + f) \cdot T + f' \cdot j_g + \frac{3}{2}f'' + g. \quad (4.49)$$

It should be noted that  $K'$  is well defined if and only if the constant term  $3/2f'' + g$  is finite. This impose a constraint for possible choice of  $f(z)$ .

#### 4.1 Perturbative vacuum

Let us first study perturbative vacuum solution. The result of section 3 for perturbative vacuum provides a hint to this problem. As similar to (3.13), we expect that (4.49) is written as a similarity transformation,

$$K' = UKU^{-1}. \quad (4.50)$$

For TT solution, it turns out that the “twisted” conformal transformation realizes this:

$$\begin{aligned} U &= \exp \left( \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} v(z) \tilde{T}(z) \right) \\ &= e^{v \cdot \tilde{T}}. \end{aligned} \quad (4.51)$$

Here  $\tilde{T}(z)$  is the twisted Virasolo generator [28, 11]

$$\tilde{T}(z) = T(z) - \partial j_g(z), \quad (4.52)$$



where  $j_g(z) =: c(z)b(z) :$  is the ghost number current. Central charge of the twisted CFT is 24 therefore an OPE between the energy momentum tensor with other operator involves an anomalous constant. A finite conformal map generated by  $v(z)$  is given by

$$y(z) = e^{v(z)\partial} z. \quad (4.53)$$

The details of a proof of (4.50) is shown in appendix A. The relation between  $f(z)$  in (4.49) and the conformal transformation  $y(z)$  is also given in appendix:

$$1 + f(y) = \frac{dy}{dz}. \quad (4.54)$$

This relation allows us to write a twisted conformal transformation of  $K$  again as a line integral in the new coordinate  $y$ . We require that the conformal map  $y(z)$  leaves imaginary axis invariant. In terms of a parameter  $t$  along the imaginary axis, this means

$$y(it)^* = -y(it). \quad (4.55)$$

We can further restrict  $y$  such that  $y(z)^* = y(z^*)$  and  $y(z) = -y(-z)$ . Then,  $y(z)$  is an odd function under the former condition.

We proceed evaluation of gauge invariant observables. For the gauge invariant overlap, we have

$$\text{Tr}_V \left[ \frac{1}{1 + K'} c \right] = \text{Tr} \left[ U^{-1} V U \frac{1}{1 + K} U^{-1} c U \right]. \quad (4.56)$$

The important aspect of the twisted conformal transformation is that  $c$  transforms as weight 0 tensor. Therefore, the  $c$  insertion in (4.56) left invariant,

$$\begin{aligned} U^{-1} c U &= U^{-1} c(0) U \\ &= c(y^{-1}(0)) \\ &= c, \end{aligned} \quad (4.57)$$

by requiring  $y(0) = y^{-1}(0) = 0$ . On the other hand, the transformation of the closed string vertex operator  $V$  involves a conformal factor since transformation of matter vertex operator  $V_X$  is not affected by the twist. We would like to impose further condition on  $y(z)$  which leaves  $V$  invariant. Since  $V$  is located at imaginary infinity, we require

$$y(\pm i\infty) = \pm i\infty, \quad y'(\pm\infty) = 1. \quad (4.58)$$

Under these conditions for  $y$ , the matter vertex operator  $V_X$  produces no conformal factor, and the trace is reduced to the Erler-Schnabl's one,

$$\text{Tr}_V \left[ \frac{1}{1 + K'} c \right] = \text{Tr}_V \left[ \frac{1}{1 + K} c \right]. \quad (4.59)$$

As similar to the case of section 3, this cancels first term of (2.6) therefore the gauge invariant overlap vanishes as expected. We note that the boundary conditions for  $y$ ,

$$y'(\pm i\infty) = 1, \quad (4.60)$$

can be translated into the boundary condition for  $f(y)$  according to (4.54) as

$$f(\pm i\infty) = 0. \quad (4.61)$$

This is the mid point condition which has been used to define TT solution [4, 26].

Evaluation of the action is more straightforward. The second trace of (2.7) is

$$\text{Tr} \left[ U \frac{1}{1+K} U^{-1} c U \frac{1}{1+K} U^{-1} c U K U^{-1} c \right] = \text{Tr} \left[ \frac{1}{1+K} c \frac{1}{1+K} c K c \right] \quad (4.62)$$

where we use the  $U$  invariance of  $c$  in right hand side. This again cancels the first term of (2.7) therefore yields vanishing action.

As an explicit example of the perturbative vacuum solution, we consider a normalized Gaussian

$$f(z) = \frac{1}{\sqrt{2\pi}s} \exp \left( \frac{z^2}{2s^2} \right), \quad (4.63)$$

where  $s$  is a real constant. It is realized that this  $f(z)$  and also  $g(z)$  dump well at imaginary infinity. The anomalous constants in  $K'$ , i.e., the last two terms (4.49) are also finite. Therefore we expect that the solution has finite contraction with wedge based states. In this example, the finite conformal map  $y(z)$  is only available as a numerical solution of the differential equation (4.54).

## 4.2 Tachyon vacuum

We next study tachyon vacuum in TT solution. Let us recall identity based solutions in  $KBc$  subalgebra we discussed in section 3. There, a crucial difference between tachyon vacuum solution and perturbative vacuum solution is the existence of isolated  $c$  term. Therefore, we consider a one parameter family such that  $g(z)$  in (4.29) approaches to delta function. In such case, the second term of (4.29) localizes to the boundary. We then begin from the normalized Gaussian  $g(z)$ :

$$g(z) = \frac{1}{\sqrt{2\pi}s} \exp \left( \frac{z^2}{2s^2} \right), \quad (4.64)$$

where  $s \rightarrow 0$  limit gives delta function. Corresponding  $f(z)$  can be obtained from the formula (4.40):

$$\omega(z) = \left( \frac{\pi}{2} \right)^{\frac{1}{4}} \sqrt{s} \cdot \text{erfi} \left( \frac{z}{2s} \right) \quad (4.65)$$

$$f(z) = -\frac{1}{4} \left( \frac{\pi}{2} \right)^{\frac{1}{2}} s \cdot \left( \text{erfi} \left( \frac{z}{2s} \right) \right)^2 - 1 \quad (4.66)$$

where  $\text{erfi}(z) = -i \text{erf}(iz)$  is the imaginary error function. An integration constant in  $\omega(z)$  is fixed so that  $f(z)$  becomes an even function. As seen from fig. 1,  $f(z)$  is very close to  $-1$  and there is a small peek around  $t = 0$ . As  $s$  become smaller, the horizontal line further approaches to  $-1$  and the peak become negligible. Thus in the  $s \rightarrow 0$  limit,

$$f(z) \rightarrow -1. \quad (4.67)$$

Therefore, the  $(1+f) \cdot T$  terms approaches to zero in this limit. On the other hand, the anomalous constant  $\frac{3}{2}f'' + g$  just becomes 1, since  $g$  contribution is just an integration of the normalized Gaussian, and  $f''$  contribution becomes zero since it is boundary values of  $f'$  which vanishes at imaginary infinity. Therefore we conclude that

$$K' \rightarrow 1 \quad (4.68)$$

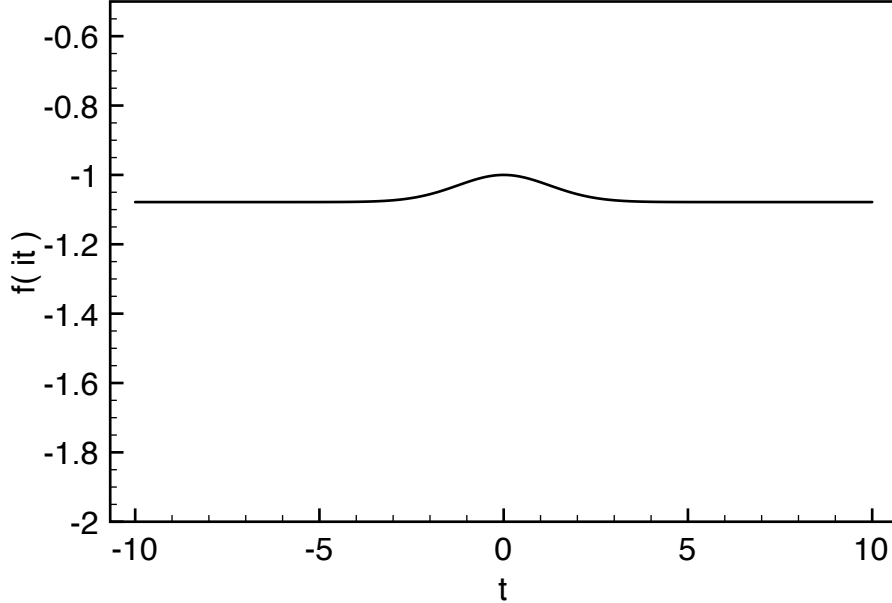


Figure 1: A plot of  $f(it)$  at  $s = 1.0$ .

in  $s \rightarrow 0$  limit. This limit just corresponds  $K' = 1$  for the tachyon vacuum solution  $c - Kc$ . Then, gauge invariant observables are evaluated in exactly same manner as in section 3. This result serves an analytic proof that particular TT scalar solution corresponds to the tachyon vacuum and successfully derives gauge invariant observables.

Finally, we note that this example cannot be written in pure gauge form derived in section 4.1. At least  $s \rightarrow 0$  limit corresponds to a singular conformal map, since (4.54) means  $y'(z) \rightarrow 0$  in this limit, therefore the image of  $z$  shrinks to a point.

## 5 Summary and discussions

We have seen that there is a special limit of TT scalar solution in which the deformed generator  $K'$  becomes constant. In such limit, gauge invariant observables can be evaluated analytically with the help of Maccaferri's formula. We also find a twisted conformal transformation which maps  $K$  to  $K'$ . This corresponds to the perturbative vacuum.

Apparently much aspects to be understood. We have not yet identified which condition for  $f(z)$  distinguish between perturbative vacuum and tachyon vacuum. It is also to be identified that whether the example given in section 4.2 for finite  $s$  corresponds to tachyon vacuum. It will require a direct evaluation of a trace which is more complicated than the marginal case.

Another important feature of the Maccaferri's formalism is the appearance of the KOS like boundary condition changing (BCC) operator. In our case of TT scalar, the corresponding BCC operator can be written  $\sigma_L = \exp(\chi_h)$  and

$$\chi_h = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} h(z) \varphi(z), \quad (5.69)$$

where  $\varphi(z)$  is the bosonized ghost. Remembering  $j_g(z) \sim \partial\varphi(z)$ , one can see the formal

resemblance of our solution to boundary deformation. Our solution looks as if it describes a “boundary” deformation by the ghost number current. Since the ghost current has singular self OPE, regularization by  $h(z)$  is required. The tachyon vacuum would be understood as a very singular limit of such “deformation”. Such interpretation of the tachyon vacuum by singular deformation will be useful tool, since it serves “boundary” CFT description for the string field theory around the tachyon vacuum, which might not exist in usual sence. In such context,  $\sigma_L$  should be understood as “boundary removing” operator. However, it is not clear whether above speculation works. The solution does not look similar to the KOS solution [29] since the ghost number current, bosonized ghost, and the BCC operator belong to the ghost sector so the solution behaves quite differently from the marginal solution made from pure matter current. It will also be interesting to apply the method developed in [30] in order to render singular ghost current OPE regular one.

## A Proof of (4.50)

In this section we prove the formula (4.50),

$$UKU^{-1} = (1 + f) \cdot T + f' \cdot j_g + \frac{3}{2}f'' - \frac{f'^2}{1 + f}, \quad (\text{A.70})$$

where  $U$  is the twisted conformal transformation:

$$U = \exp \left( \oint dz v(z) \tilde{T}(z) \right). \quad (\text{A.71})$$

In order to evaluate left hand side of (A.71), we divide  $K$  as

$$K = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} T'(z) + \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \partial j_g(z) \quad (\text{A.72})$$

and evaluate each term. Relevant finite transformations are [31]

$$U\tilde{T}(z)U^{-1} = y'^2\tilde{T}(y) + \frac{c}{12} \left[ \frac{y'''}{y'} - \frac{3}{2} \left( \frac{y''}{y'} \right)^2 \right], \quad (\text{A.73})$$

$$U\partial j_g(z)U^{-1} = y''j_g(y) + y'^2\partial j_g(y) + \frac{3-2\beta}{2} \left[ \frac{y'''}{y'} - \left( \frac{y''}{y'} \right)^2 \right], \quad (\text{A.74})$$

where a prime denote derivative with respect to  $z$ .  $c$  is the central charge and  $\beta$  is the parameter that specifies conformal weights of ghosts. A sum of (A.73) and (A.74) gives a finite transformation of  $T(z)$ :

$$\begin{aligned} UT(z)U^{-1} &= y'^2\tilde{T}(y) + y''j_g(y) + y'^2\partial j_g(y) + \left( \frac{c}{12} + \frac{3-2\beta}{2} \right) \frac{y'''}{y'} + \left( -\frac{2}{3} \cdot \frac{c}{12} - \frac{3-2\beta}{2} \right) \left( \frac{y''}{y'} \right)^2 \\ &= y'^2T(y) + y''j_g(y) + \frac{3}{2} \frac{y'''}{y'} - \frac{5}{2} \left( \frac{y''}{y'} \right)^2. \end{aligned} \quad (\text{A.75})$$

We have applied  $c = 24$ ,  $\beta = 2$  and  $\tilde{T}(z) = T(z) - \partial j_g(z)$  in the last line. Then an integral of (A.75) with respect to  $z$  gives transformation of  $K$ .

$$UKU^{-1} = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \left( y'^2T(y) + y''j_g(y) + \frac{3}{2} \frac{y'''}{y'} - \frac{5}{2} \left( \frac{y''}{y'} \right)^2 \right) \quad (\text{A.76})$$

Next, we would like to change variable  $z$  to  $y$  in the integral (A.76), with assuming that  $y$  leave imaginary axis unchanged. Then, it turns out that the first term of (A.75) matches with  $f \cdot T$  in (A.70) if we assign

$$y' = 1 + f(y). \quad (\text{A.77})$$

The constant term of (A.75) can be written in terms of  $f$  by applying formulas obtained by differentiating (A.77):

$$y'' = y' f'(y), \quad y''' = y'' f'(y) + y'^2 f''(y). \quad (\text{A.78})$$

Then we have

$$UKU^{-1} = \int_{-i\infty}^{i\infty} \frac{dy}{2\pi i} \left( (1 + f(y))T(y) + f'(y)j_g(y) + \frac{3}{2}f''(y) - \frac{f'(y)^2}{1 + f(y)} \right), \quad (\text{A.79})$$

which coincides with the expression for  $K'$  given in (A.70).

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